

EXPLICIT PRESENTATIONS OF NONSPECIAL LINE BUNDLES AND SECANT SPACES

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ABSTRACT. A line bundle \mathcal{L} on a smooth curve X is nonspecial if and only if \mathcal{L} admits a presentation $\mathcal{L} \simeq \mathcal{K}_X - D + E$ for some divisors $D \geq 0$, $E > 0$ on X with $\gcd(D, E) = 0$ and $h^0(X, \mathcal{O}_X(D)) = 1$. In this work, we define a minimal presentation of \mathcal{L} which is minimal with respect to $\deg E$ among the presentations. If $\mathcal{L} \simeq \mathcal{K}_X - D + E$ with $\deg E \geq 3$ is a minimal, then \mathcal{L} is very ample and any q -points of $\varphi_{\mathcal{L}}(X)$ with $q \leq \deg E - 1$ are in general position but the points of $\varphi_{\mathcal{L}}(E)$ are not. We investigate sufficient conditions on divisors D, E for $\mathcal{L} \simeq \mathcal{K}_X - D + E$ to be minimal. Through this, for a number n in some range, it is possible to construct a nonspecial very ample line bundle $\mathcal{L} \simeq \mathcal{K}_X - D + E$ on X with/without an n -secant $(n - 2)$ -plane of the embedded curve by taking divisors D, E on X . As its applications, we construct nonspecial line bundles which show the sharpness of Green and Lazarsfeld's Conjecture on property (N_p) for general n -gonal curves and simple multiple coverings of smooth plane curves.

1. Introduction

Throughout this paper, we mean a curve by a reduced irreducible algebraic curve over an algebraically closed field of characteristic zero. We will investigate properties of nonspecial line bundles \mathcal{L} on a smooth curve X with respect to presentations such as $\mathcal{L} \simeq \mathcal{K}_X - D + E$ by using the canonical line bundle \mathcal{K}_X and effective divisors D, E on X with $\gcd(D, E) = 0$ and $h^0(X, \mathcal{O}_X(D)) = 1$.

To an arbitrary pair of effective divisors D, E on X with $\gcd(D, E) = 0$ we can associate a line bundle $\mathcal{L} \simeq \mathcal{K}_X - D + E$, which is nonspecial if $h^0(X, \mathcal{O}_X(D)) = 1$ and $E > 0$. Conversely, a nonspecial line bundle \mathcal{L} on X also admits an equivalence $\mathcal{L} \simeq \mathcal{K}_X - D + E$, which will be called a presentation of \mathcal{L} , for some $D \geq 0$, $E > 0$ with $\gcd(D, E) = 0$ and $h^0(X, \mathcal{O}_X(D)) = 1$. However, a nonspecial line bundle may have several different presentations. Thus we define a minimal

2010 *Mathematics Subject Classification.* 14C20, 14E25, 14H30, 16E05.

Key words and phrases. algebraic curve, nonspecial line bundle, nonspecial divisor, minimal presentation, q -very ample, order of very ampleness, secant space, multiple covering, gonality, property (N_p) , extremal line bundle.

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (2011-0011224).

presentation(:minimal with respect to $\deg E$) as the most efficient one in some sense.

Assume that \mathcal{L} is minimally presented by $\mathcal{K}_X - D + E$. If $\deg E \geq 3$, then \mathcal{L} is very ample and any q -points of $\varphi_{\mathcal{L}}(X)$ with $q \leq \deg E - 1$ are in general position but the points of $\varphi_{\mathcal{L}}(E)$ are not(see Proposition 2.2). Accordingly, nonspecial line bundles can be distinguished by their minimal presentations. Thus finding sufficient conditions for minimality can be a major issue in this study. In Section 3, we explore some sufficient conditions for such presentations to be minimal on multiple coverings. Note that every smooth curve is a multiple covering of \mathbb{P}^1 .

Now, consider some details of the brief outline above. Let X be a smooth curve of genus $g \geq 2$ and \mathcal{L} be a line bundle on X . If \mathcal{L} is special(: $h^1(X, \mathcal{L}) > 0$), then its residual line bundle $\mathcal{K}_X \otimes \mathcal{L}^{-1}$ plays a role in investigating the properties of \mathcal{L} or X itself, since $\mathcal{K}_X \otimes \mathcal{L}^{-1}$ has global sections and is associated to an effective divisor. On the other hand, if \mathcal{L} is nonspecial(: $h^1(X, \mathcal{L}) = 0$) then the residual line bundle $\mathcal{K}_X \otimes \mathcal{L}^{-1}$ has no global sections and hence no corresponding effective divisors. Accordingly, it is a natural analyzing approach to express $\mathcal{K}_X \otimes \mathcal{L}^{-1}$ in terms of effective divisors as follows.

Let \mathcal{L} be a nonspecial line bundle on a smooth curve X of genus g . Then, there exists a divisor $E > 0$ such that

$$h^0(X, \mathcal{K}_X \otimes \mathcal{L}^{-1}(E)) = 1, \quad h^0(X, \mathcal{K}_X \otimes \mathcal{L}^{-1}(E')) = 0 \text{ for } E' < E.$$

Hence we have an effective divisor D such that

$$\mathcal{K}_X \otimes \mathcal{L}^{-1}(E) \simeq \mathcal{O}_X(D), \text{ equivalently, } \mathcal{L} \simeq \mathcal{K}_X(-D + E),$$

with $h^0(X, \mathcal{O}_X(D)) = 1$ and $\gcd(D, E) = 0$.

For an efficient description, we will use some notations as the following:

- (i) g_d^0 : an effective divisor of degree d with $h^0(X, \mathcal{O}_X(g_d^0)) = 1$,
- (ii) (D, E) : the greatest common divisor of divisors D and E ,
- (iii) $\mathcal{L} - g_d^n$: the line bundle $\mathcal{L}(-D)$ where $|D| = g_d^n$.

Using these, a nonspecial line bundle \mathcal{L} can be written as

$$\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + E$$

for some divisors g_d^0 and $E > 0$ on X with $(g_d^0, E) = 0$. Conversely, if $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + E$ for such g_d^0 and $E > 0$ then \mathcal{L} is nonspecial.

Definition 1.1. Let \mathcal{L} be a nonspecial line bundle on X . (1) If $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + E$ with $(g_d^0, E) = 0$ and $E > 0$ then the equivalence is called a presentation of type (d, e) , where $e := \deg E$. (2) $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + E$ is said to be minimal if any presentation $\mathcal{L} \simeq \mathcal{K}_X - g_t^0 + F$ satisfies $\deg F \geq \deg E$. (3) A presentation of type $(0, e)$, i.e., $\mathcal{L} \simeq \mathcal{K}_X + E$ is said to be trivial.

Assume that a nonspecial line bundle \mathcal{L} is presented by $\mathcal{K}_X - g_d^0 + E$. Then we have the equality $h^0(X, \mathcal{L}) - h^0(X, \mathcal{L}(-E)) = \deg E - 1$. Accordingly, \mathcal{L} is not

globally generated if $\deg E = 1$, and \mathcal{L} is not very ample if $\deg E = 2$. Hence it is a natural question whether a nonspecial line bundle $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + E_3$ with $\deg E_3 = 3$ is very ample or not. If the presentation $\mathcal{K}_X - g_d^0 + E_3$ is not minimal, equavelently, \mathcal{L} has another presentation $\mathcal{L} \simeq \mathcal{K}_X - g_{\leq d-1}^0 + E'$ with $\deg E' \leq 2$, then \mathcal{L} is not very ample.

Likewise, a given nonspecial line bundle \mathcal{L} may admit various presentations. Here, the degrees of g_d^0 and E as well as the divisors g_d^0 and E depend on presentations of \mathcal{L} . However, a special line bundle \mathcal{L} can be written as $\mathcal{K}_X - D$ for $D \in |\mathcal{K}_X \otimes \mathcal{L}^{-1}|$ which is unique up to linear equivalence. Thus we would be naturally interested in a minimal presentation and its uniqueness.

Assume that a nonspecial line bundle \mathcal{L} is minimally presented by $\mathcal{K}_X - g_d^0 + E$ with $\deg E \geq 3$. Then, by the Riemann-Roch Theorem \mathcal{L} is very ample and the embedded curve $\varphi_{\mathcal{L}}(X)$ admits a $\deg E$ -secant $(\deg E - 2)$ -plane $\langle E \rangle_{\mathcal{L}}$ but does not admit n -secant $(n - 2)$ -planes for any $n \leq \deg E - 1$. Here, $\langle E \rangle_{\mathcal{L}} := \cap \{H \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \mid H \cdot \varphi_{\mathcal{L}}(X) \geq E\}$, $\mathbb{P} := \mathbb{P}H^0(X, \mathcal{L})^*$. Moreover, if $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + E$ is a unique minimal presentation then $\langle E \rangle_{\mathcal{L}}$ is a unique $\deg E$ -secant $(\deg E - 2)$ -plane of $\varphi_{\mathcal{L}}(X)$.

Now, observe the case that a nonspecial line bundle \mathcal{L} is trivially presented by $\mathcal{K}_X + E$. This $\mathcal{L} \simeq \mathcal{K}_X + E$ is in itself a minimal presentation and the family of such presentations of \mathcal{L} corresponds to the linear system $|E|$. Note that the minimal presentations of nonspecial line bundles \mathcal{L} with $\deg \mathcal{L} \geq 3g - 2$ are always trivial (see Proposition 2.2, (vi)), whereas every presentation of a nonspecial globally generated line bundle \mathcal{L} with $\deg \mathcal{L} \leq 2g - 1$ is nontrivial, i.e., \mathcal{L} is always presented by $\mathcal{K}_X - g_d^0 + E$ with $g_d^0 \neq 0$ (see Remark 2.3, (ii)).

Assume that \mathcal{L} admits a nontrivial presentation $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + E$. Then we may assume that $h^0(X, \mathcal{O}_X(E)) = 1$ (see Proposition 2.2, (v)) and hence denote the divisor E by ξ_e^0 , $e := \deg E$. Accordingly, \mathcal{L} can be written as

$$\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0 \quad \text{with } (g_d^0, \xi_e^0) = 0$$

which is a better explicit description than the type of $\mathcal{L} \simeq \mathcal{K}_X - D + E$, since notations such as g_d^0 , ξ_e^0 include information on degrees and dimensions of $|D|$ and $|E|$.

Conversely, we obtain a nonspecial line bundle $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + E$ on X whenever we take effective divisors g_d^0 and $E \geq 0$ on X with $(g_d^0, E) = 0$. If we obtain sufficient conditions on D, E for $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + E$ to be minimal, then we can construct a nonspecial line bundle $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + E$ with/without an n -secant $(n - 2)$ -plane of the embedded curve $\varphi_{\mathcal{L}}(X)$ by taking some divisors D, E on X . This study could also provide a clue to detect the family of nonspecial line bundles with specific properties for such secant spaces.

Note that for a nonspecial very ample $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + E$ the curve $\varphi_{\mathcal{L}}(X)$ is a projection of $\varphi_{\mathcal{K}_X + E}(X)$ from $\langle g_d^0 \rangle_{\mathcal{K}_X + E}$, whereas for a special very ample

line bundle $\mathcal{L} \simeq \mathcal{K}_X - D$ on a nonhyperelliptic curve X the curve $\varphi_{\mathcal{L}}(X)$ is a projection of the canonical curve $\varphi_{\mathcal{K}_X}(X)$ from $\langle D \rangle_{\mathcal{K}_X}$. This gives another perspective on our study that finding a minimal presentation of a very ample line bundle \mathcal{L} is equivalent to choosing a minimal effective divisor E satisfying (1) $\varphi_{\mathcal{L}}(X)$ is a projection of $\varphi_{\mathcal{K}_X+E}(X)$, (2) both $\varphi_{\mathcal{L}}(X)$ and $\varphi_{\mathcal{K}_X+E}(X)$ possess the same properties with respect to n -secant $(n-2)$ -spaces.

It is interesting that every presentation $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ with $d+e \leq \text{gon}(X)$ (resp. $d+e < \text{gon}(X)$) is a (resp. unique) minimal one (see Theorem 2.8). On the other hand, there are examples of $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ with $d+e \geq \text{gon}(X) + 1$ which are not minimal (see Example 2.11).

Furthermore, if \mathcal{L} admits a presentation $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ with $d+e = \text{gon}(X)$, then the number of presentations of \mathcal{L} with the same type (d, e) is at most one plus the number of pencils $g_{\text{gon}(X)}^1$ on X (see Remark 2.10, (iii)). This means that for $e \geq 3$ the number of e -secant $(e-2)$ -planes of $\varphi_{\mathcal{L}}(X)$ is at most one plus the number of pencils $g_{\text{gon}(X)}^1$. In addition, we show that an m -fold covering X of an elliptic curve with $\text{gon}(X) = 2m$ have a line bundle $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ admitting infinitely many presentations of the same type (d, e) with $d+e = \text{gon}(X)$ (see Example 2.12).

Note that for a smooth curve X with a well known g_d^r the line bundles $\mathcal{L} \simeq \mathcal{K}_X - g_d^r + \xi_e^0$ are very typical nonspecial line bundles on X . Thus we investigate minimal presentations of $\mathcal{L} \simeq \mathcal{K}_X - g_d^r + \xi_e^0$ on the curve X . To do this, we set $\beta := \max\{\deg(\xi_e^0, D) \mid D \in g_d^r\}$. Then we may expect the minimality of $\mathcal{L} \simeq \mathcal{K}_X - g_{d-\beta}^0 + \xi_{e-\beta}^0$, where $\deg(\xi_e^0, D) = \beta$ for a $D \in g_d^r$, $g_{d-\beta}^0 := g_d^r - (\xi_e^0, D)$ and $\xi_{e-\beta}^0 := \xi_e^0 - (\xi_e^0, D)$. Such an expectation holds under some specific conditions and there is also an example where the expectation fails (see Theorem 2.14, Example 2.15).

In section 3, we investigate sufficient conditions for the minimality of presentations of nonspecial line bundles on multiple coverings. For an n -fold covering X via $\phi : X \rightarrow Y$ a presentation $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ with $d+e \leq \mu$ is minimal if $\deg(g_d^0, \phi^*(P)) + \deg(\xi_e^0, \phi^*(Q)) \leq n$ for any $P, Q \in Y$, where $\mu := \min\{\deg \mathcal{N} \mid \mathcal{N} : \text{globally generated and not composed with } \phi\}$ (see Theorem 3.2). Specifically, the number μ is greater than $\frac{g+1}{2}$ (resp. $\frac{g-n\gamma}{n-1}$) for a general n -gonal curve (resp. for a simple n -fold covering of a smooth curve of genus γ). Here, a multiple covering is said to be *simple* if the covering morphism does not factor through. Note that general g_d^0 and ξ_e^0 on X satisfy the condition $\deg(g_d^0, \phi^*(P)) + \deg(\xi_e^0, \phi^*(Q)) \leq n$ for any $P, Q \in Y$. Thus whenever we take general g_d^0 and ξ_e^0 on a multiple covering X with $e \geq 2$ and $d+e \leq \mu$, we obtain a nonspecial line bundle $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ on X which is $(e-2)$ -very ample. This means that for any positive number $q \leq \mu - 1$ we can construct q -very ample nonspecial line bundles on X with a given degree $\geq 2g - 1 + 2e - \mu$.

It is also notable that for an n -fold covering $\phi : X \rightarrow \mathbb{P}^1$ the condition such that $\deg(g_d^0, \phi^*(P)) + \deg(\xi_e^0, \phi^*(Q)) \leq n$ for any $P, Q \in \mathbb{P}^1$ is necessary for $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ to be minimal (see Proposition 3.1).

We also deal with minimal presentations of typical line bundles such as $\mathcal{L} \simeq \mathcal{K}_X - \phi^*(g_d^2) + \xi_{e+2}^0$ and $\mathcal{M} \simeq \mathcal{K}_X - \phi^*(g_{d-1}^1) + \xi_{e+1}^0$ on a simple n -fold covering X of a smooth plane curve Y via $\phi : X \rightarrow Y$ (see Theorem 3.10).

In section 4, we apply minimal presentations of nonspecial line bundles to investigate property (N_p) , since $(p+1)$ -very ampleness is very closely connected with property (N_p) . M. Green and R. Lazarsfeld showed that a line bundle \mathcal{L} of degree $2g+p$ on a nonhyperelliptic curve X satisfies (N_p) if and only if $\varphi_{\mathcal{L}}(X)$ has no $(p+2)$ -secant p -planes, that is, \mathcal{L} is $(p+1)$ -very ample (see [8], Theorem 2). On the other hand, it is well known that if a very ample line bundle \mathcal{L} on X fails to be $(p+1)$ -very ample then \mathcal{L} does not satisfy (N_p) .

Along this line, the validity of its converse under the condition $\deg \mathcal{L} \geq 2g+1+p-2h^1(X, \mathcal{L}) - \text{Cliff}(X)$ was conjectured by M. Green and R. Lazarsfeld in [7]. It is called Green-Lazarsfeld's conjecture on (N_p) . In fact, they have shown in the paper that this conjecture holds for (N_0) . Since M. Aprodu demonstrated in [3] that general gonality curves satisfy Green's Conjecture on syzygies of canonical curves (this validity was remarked after Theorem 2 in [3]), we can easily see that the special line bundles on them satisfy Green-Lazarsfeld's conjecture on (N_p) by Theorem 1 in [6]. Thus a natural question is on the existence of a very ample line bundle \mathcal{L} on X with $\deg \mathcal{L} = 2g+p-2h^1(X, \mathcal{L}) - \text{Cliff}(X)$ which does not satisfy (N_p) even if $\varphi_{\mathcal{L}}(X)$ does not admit a $(p+2)$ -secant p -plane. Such a line bundle will be called *an extremal line bundle for Green-Lazarsfeld's conjecture on (N_p)* .

Using theorems on the minimality of presentations in section 3, we verify that general n -gonal curves and simple n -fold coverings of smooth plane curves carry nonspecial extremal line bundles for Green-Lazarsfeld's conjecture on (N_p) (see Theorem 4.5, 4.8). To do this study, we compute the Clifford index of multiple coverings of smooth plane curves (see Proposition 4.7).

2. The presentations of nonspecial line bundles

In this section, we investigate properties of presentations of nonspecial line bundles on smooth curves. This study naturally focuses on the minimal presentations of nonspecial line bundles which can be regarded as efficient ones. Before going to this observation, we will consider a type of refinement of very ampleness which is closely related to minimal presentation.

Recall that a line bundle \mathcal{L} on a smooth curve X is said to be q -very ample, $q \geq 0$, if $h^0(X, \mathcal{L}) - h^0(X, \mathcal{L}(-F)) = \deg F$ for any effective divisor F with $\deg F \leq q+1$. Specifically, 0-very ampleness and 1-very ampleness mean globally generatedness and very ampleness, respectively. If $q \geq 1$, then \mathcal{L} is very ample and the embedded curve $\varphi_{\mathcal{L}}(X) \subseteq \mathbb{P}H^0(X, \mathcal{L})^*$ has no n -secant $(n-2)$ -planes for

any number $n \leq q + 1$, equivalently, $\dim \langle F \rangle_{\mathcal{L}} = \deg F - 1$ for any effective divisor F on X with $\deg F \leq q + 1$. Now, we define an invariant to measure the linear position property of $\varphi_{\mathcal{L}}(X)$ in $\mathbb{P}H^0(X, \mathcal{L})^*$.

Definition 2.1. *The order of very ampleness of a line bundle \mathcal{L} is defined by*

$$\text{Ova}(\mathcal{L}) := \max\{ q \in \mathbf{Z}^{\geq 0} \mid \mathcal{L} \text{ is } q\text{-very ample} \}.$$

In the following theorem, we examine basic properties of presentations of non-special line bundles.

Proposition 2.2. *Let $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + E$ be a presentation of a nonspecial line bundle \mathcal{L} on a smooth curve X of genus $g \geq 2$. Then we have the following.*

- (i) $\mathcal{L} \simeq \mathcal{K}_X - g_t^0 + F$ for a $F \geq 0$ with $\deg F < \deg E$ if $g_d^0 \neq 0$, $h^0(X, \mathcal{O}_X(E)) \geq 2$.
- (ii) $d \geq \deg E - 1$ if $\deg \mathcal{L} \leq 2g - 1$.
- Specifically, if $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + E$ is a minimal presentation then*
 - (iii) $\deg F \geq \deg E$ in case $h^0(X, \mathcal{L}) - h^0(X, \mathcal{L}(-F)) \leq \deg F - 1$,
 - (iv) $\text{Ova}(\mathcal{L}) = \deg E - 2$,
 - (v) $h^0(X, \mathcal{O}_X(E)) = 1$ in case $d > 0$,
 - (vi) $d = 0$ in case $\deg \mathcal{L} \geq 3g - 2$,
 - (vii) $d > 0$ and $h^0(X, \mathcal{O}_X(E)) = 1$ in case $\deg E \geq 2$ and $\deg \mathcal{L} \leq 2g - 1$,
 - (viii) $d \leq g - 1$; and $d \leq g - 2$ in case $\deg E \geq 3$.

Proof. (i) Assume that $h^0(X, \mathcal{O}_X(E)) \geq 2$. For any $P \in \text{supp}(g_d^0)$ there is an effective divisor $E' \simeq E$ with $(g_d^0, E') \geq P$. Set $g_t^0 := g_d^0 - (g_d^0, E')$ and $F := E' - (g_d^0, E')$. Then there is another presentation $\mathcal{L} \simeq \mathcal{K}_X - g_t^0 + F$ with $\deg F < \deg E$.

(ii) For $\deg \mathcal{L} \leq 2g - 1$, the equality $\deg \mathcal{L} = 2g - 2 - d + \deg E$ gives $d \geq \deg E - 1$.

(iii) Assume that $h^0(X, \mathcal{L}) - h^0(X, \mathcal{L}(-F)) \leq \deg F - 1$ for a divisor F on X . Then the Riemann-Roch Theorem gives the inequality $h^0(X, \mathcal{K}_X \otimes \mathcal{L}^{-1}(F)) \geq 1$, which implies $\deg F \geq \deg E$ by the minimality of $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + E$.

(iv) This result follows from (iii) and the equality $h^0(X, \mathcal{L}) - h^0(X, \mathcal{L}(-E)) = \deg E - 1$.

(v) This is trivial by (i)

(vi) The condition $\deg \mathcal{L} \geq 3g - 2$ yields $\deg \mathcal{L} \otimes \mathcal{K}_X^{-1} \geq g$ and so there is an effective divisor E such that $\mathcal{L} \otimes \mathcal{K}_X^{-1} \simeq \mathcal{O}_X(E)$, equivalently, $\mathcal{L} \simeq \mathcal{K}_X + E$, which is in itself a minimal presentation.

(vii) Using (ii) and (v), we get $d > 0$ and $h^0(X, \mathcal{O}_X(E)) = 1$ in case $\deg E \geq 2$ and $\deg \mathcal{L} \leq 2g - 1$.

(viii) Set $r := h^0(X, \mathcal{L}) - 1$. Choose a $F \leq G \in |\mathcal{L}|$ with $\deg F = r + 1$. Then the divisor F satisfies $h^0(X, \mathcal{L}) - h^0(X, \mathcal{L}(-F)) \leq \deg F - 1$, which yields $\deg E \leq \deg F = r + 1$ by (iii). Hence, we obtain $d \leq g - 1$ since $r = (2g - 2 - d + \deg E) - g$.

Assume $\deg E \geq 3$. Then \mathcal{L} is very ample. Since $r = \deg \mathcal{L} - g$, the condition $g \geq 2$ gives $\deg \varphi_{\mathcal{L}}(X) \geq r + 2$, whence the smooth curve $\varphi_{\mathcal{L}}(X)$ has a r -secant

$(r - 2)$ -plane by Lemma in [12]. By (iii), we have

$$\deg E \leq r = (2g - 2 - d + \deg E) - g,$$

which implies $d \leq g - 2$. Thus the result (viii) is verified. \square

Remark 2.3. (i) $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + E$ is a minimal presentation if and only if $\text{Ova}(\mathcal{L}) = \deg E - 2$.

(ii) The minimal presentations of nonspecial line bundles \mathcal{L} with $\deg \mathcal{L} \geq 3g - 2$ are always trivial. On the other hand, all the minimal presentations of globally generated nonspecial line bundles \mathcal{L} with $\deg \mathcal{L} \leq 2g - 1$ are nontrivial since $\deg E \geq 2$ by being globally generated.

(iii) To arbitrary pair of effective divisors g_d^0 and ξ_e^0 with $(g_d^0, \xi_e^0) = 0$, we can associate a nonspecial line bundle $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$. By Proposition 2.2, (vii), it is enough to consider the divisors g_d^0 only in the range $d \leq g - 1$ (resp. $d \leq g - 2$) for such construction of (resp. very ample) nonspecial line bundles.

(iv) If \mathcal{L} is minimally presented by $\mathcal{K}_X - g_d^0 + E$ with $\deg E \geq 3$, then the embedded curve $\varphi_{\mathcal{L}}(X)$ has no n -secant $(n - 2)$ -planes for $n \leq \deg E - 1$. Moreover, if $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + E$ is a unique minimal presentation, then $\langle E \rangle_{\mathcal{L}}$ is a unique $\deg E$ -secant $(\deg E - 2)$ -plane.

The following theorem plays a basic role in dealing with presentations of nonspecial line bundles.

Theorem 2.4. Assume that a nonspecial line bundle \mathcal{L} on a smooth curve X has two different presentations $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + E$ and $\mathcal{L} \simeq \mathcal{K}_X - g_t^0 + F$. Then $g_d^0 + F \simeq g_t^0 + E$ but $g_d^0 + F \neq g_t^0 + E$ as divisors. In particular, we have $h^0(X, \mathcal{O}_X(g_d^0 + F)) \geq 2$.

Proof. The equivalences $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + E \simeq \mathcal{K}_X - g_t^0 + F$ imply that

$$g_d^0 + F \simeq g_t^0 + E.$$

Assume that $g_d^0 + F = g_t^0 + E$ as divisors. Then we get

$$g_t^0 \leq g_d^0, \quad F \leq E,$$

according to the condition $(g_t^0, F) = 0$ for the presentation $\mathcal{L} \simeq \mathcal{K}_X - g_t^0 + F$. The condition $(g_d^0, E) = 0$ also gives

$$g_d^0 \leq g_t^0, \quad E \leq F.$$

It is a contrary to the assumption that $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + E$ and $\mathcal{L} \simeq \mathcal{K}_X - g_t^0 + F$ are different. Hence, we have $g_d^0 + F \neq g_t^0 + E$ as divisors which implies $h^0(X, \mathcal{O}_X(g_d^0 + F)) \geq 2$. Thus the theorem is proved. \square

Corollary 2.5 (Lemma 6, [9]). Let \mathcal{L} be a nonspecial line bundle on a smooth curve X which is presented by $\mathcal{K}_X - g_d^0 + E$ with $\deg E \geq 3$. If \mathcal{L} is not very

ample, then there are $g_t^0 \geq 0$ and $P, Q \in X$ such that $g_t^0 + E \simeq g_d^0 + P + Q$ and $g_t^0 + E \neq g_d^0 + P + Q$ as divisors which implies $h^0(X, \mathcal{O}_X(g_d^0 + P + Q)) \geq 2$.

As we have seen, admitting a presentation $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + E$ with $\deg E \geq 3$ does not guarantee the very ampleness of \mathcal{L} . Thus we investigate sufficient conditions for the very ampleness of $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + E$ with $d > 0$ and $\deg E \geq 3$ in the following. Here, we consider only the case of $h^0(X, \mathcal{O}_X(E)) = 1$ due to Proposition 2.2, (i).

Theorem 2.6. *Let X be a smooth curve of genus $g \geq 4$. And let g_d^0, ξ_e^0 be general effective divisors on X with $e \geq 3$ and $(g_d^0, \xi_e^0) = 0$.*

- (i) *If X is nonhyperelliptic, then $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ with $d \leq g - 3$ is very ample.*
- (ii) *If X is hyperelliptic, then $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ with $d \leq g - 2$ is very ample.*

Proof. Assume that $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ is not very ample. Corollary 2.5 gives

$$\mathcal{O}_X(g_t^0 + \xi_e^0) \simeq \mathcal{O}_X(g_d^0 + P_1 + P_2) \in W_{d+2}^\alpha(X), \quad \alpha \geq 1,$$

for some $P_1, P_2 \in X$ and g_t^0 on X , where

$$W_{d+2}^\alpha(X) := \{\mathcal{L} \in J(X) \mid h^0(X, \mathcal{L}) \geq \alpha + 1, \deg \mathcal{L} = d + 2\}.$$

- (i) Let X be a nonhyperelliptic curve. Due to the general choice of g_d^0 with $d \leq g - 3$ and H. Martens' Theorem (:(5.1) Theorem in [1]), we obtain

$$d - \alpha \leq \dim W_{d+2}^\alpha(X) \leq (d + 2) - 2\alpha - 1,$$

whence $\dim W_{d+2}^\alpha(X) = (d + 2) - 2\alpha - 1$ and $\dim |g_d^0 + P_1 + P_2| = \alpha = 1$. According to Mumford's Theorem (:(5.2) Theorem in [1]) we have one of the following cases with a base locus B :

(Case 1) $\phi : X \xrightarrow{3:1} \mathbb{P}^1$ and $|\mathcal{O}_X(g_d^0 + P_1 + P_2)| = g_3^1 + B$.

(Case 2) $\phi : X \xrightarrow{2:1} \Gamma$ for an elliptic curve Γ and $|\mathcal{O}_X(g_d^0 + P_1 + P_2)| = \phi^* g_2^1 + B$.

(Case 3) X is a smooth plane quintic and $|\mathcal{O}_X(g_d^0 + P_1 + P_2)| = g_4^1 + B$.

First, consider (Case 3). Note that every divisor of g_4^1 on X is cut out by a line in P^2 . By the general choices of g_d^0 and ξ_e^0 we obtain

$$B = g_{d-2}^0 \text{ and } B \geq \xi_{e-2}^0$$

for some $g_{d-2}^0 < g_d^0$ and $\xi_{e-2}^0 < \xi_e^0$, since $|\mathcal{O}_X(g_d^0 + P_1 + P_2)| = |\mathcal{O}_X(\xi_e^0 + g_t^0)| = g_4^1 + B$. This implies that $\xi_{e-2}^0 \leq (g_d^0, \xi_e^0) = 0$, which is contrary to $e \geq 3$.

Also (Case 2) cannot happen by the following. The general choices of g_d^0 and ξ_e^0 imply that $\deg(g_d^0, \phi^*(Q)) \leq 1$ and $\deg(\xi_e^0, \phi^*(Q)) \leq 1$ for any $Q \in \Gamma$. Thus we obtain

$$B = g_{d-2}^0, \quad B \geq \xi_{e-2}^0$$

for some $g_{d-2}^0 < g_d^0$ and $\xi_{e-2}^0 < \xi_e^0$, since $|\mathcal{O}_X(g_d^0 + P_1 + P_2)| = |g_t^0 + \xi_e^0| = \phi^* g_2^1 + B$. This cannot occur for $(g_d^0, \xi_e^0) = 0$ with $e \geq 3$ as in (Case 3).

Finally, we are led to (Case 1). Since $|\mathcal{O}_X(g_d^0 + P_1 + P_2)| = |\mathcal{O}_X(\xi_e^0 + g_t^0)| = g_3^1 + B$, the general choices of g_d^0 and ξ_e^0 give

$$B = g_{d-1}^0 \quad \text{and} \quad B \geq \xi_{e-1}^0$$

for some $g_{d-1}^0 \leq g_d^0$ and $\xi_{e-1}^0 \leq \xi_e^0$. This is a contradiction to $(g_d^0, \xi_e^0) = 0$ with $e \geq 3$. As a consequence, the result (i) is valid.

(ii) Let X be hyperelliptic. Due to the condition $d \leq g - 2$, the linear system $|\mathcal{O}_X(g_d^0 + P_1 + P_2)| = g_{d+2}^\alpha$ with $\alpha \geq 1$ is special, and hence

$$|\mathcal{O}_X(g_d^0 + P_1 + P_2)| = |\mathcal{O}_X(\xi_e^0 + g_t^0)| = \alpha g_2^1 + B.$$

Since g_d^0 is generally chosen, only two cases $\alpha = 1$ and $\alpha = 2$ can occur. According to the general choices of g_d^0 and ξ_e^0 , if $\alpha = 1$ then

$$B = g_{d-1}^0 + P_i \quad \text{and} \quad \xi_{e-1}^0 \leq B \quad \text{for some} \quad g_{d-1}^0 \leq g_d^0, \quad \xi_{e-1}^0 \leq \xi_e^0;$$

if $\alpha = 2$ then

$$B = g_{d-2}^0 \quad \text{and} \quad \xi_{e-2}^0 \leq B \quad \text{for some} \quad g_{d-2}^0 \leq g_d^0, \quad \xi_{e-2}^0 \leq \xi_e^0.$$

These are impossible for $(g_d^0, \xi_e^0) = 0$ with $e \geq 3$. Consequently, the line bundle $\mathcal{K}_X - g_d^0 + \xi_e^0$ is very ample and hence the theorem is proved. \square

Remark 2.7. Let X be a smooth curve of genus $g \geq 11$ and \mathcal{L} be a line bundle presented by $\mathcal{K}_X - g_d^0 + \xi_e^0$ for general g_d^0 and ξ_e^0 with $d \leq g - 7$ and $e \geq 4$. Using Keem's Theorem in [10] which generalizes H. Martens' Theorem, we can similarly verify that the embedded curve $\varphi_{\mathcal{L}}(X)$ has no 4-secant plane unless X is either hyperelliptic, trigonal, elliptic-hyperelliptic, a 4-sheeted covering of \mathbb{P}^1 , or a 2-sheeted covering of a curve of genus 2.

Theorem 2.8. Let \mathcal{L} be a nonspecial line bundle on a smooth curve X which admits a nontrivial presentation $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$.

- (i) If $d + e \leq \text{gon}(X)$, then $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ is a minimal presentation.
- (ii) If $d + e < \text{gon}(X)$, then $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ is a unique minimal presentation.
- (iii) If $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ with $d + e = \text{gon}(X)$ admits other presentations $\mathcal{L} \simeq \mathcal{K}_X - h_{d,j}^0 + \zeta_{e,j}^o$ of type (d, e) for $j \in J$, then all the pencils $|\mathcal{O}_X(g_d^0 + \zeta_{e,j}^o)|$ ($= |\mathcal{O}_X(h_{d,j}^0 + \xi_e^0)|$) with $j \in J$ are mutually distinct $g_{\text{gon}(X)}^1$ on X .

Proof. (i) If $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ is not a minimal presentation, then there is a ξ_s^0 with $s < e$ such that $h^0(X, \mathcal{O}_X(g_d^0 + \xi_s^0)) \geq 2$ by Theorem 2.4. It cannot occur for $d + e \leq \text{gon}(X)$.

(ii) In case $d + e < \text{gon}(X)$, Theorem 2.4 also implies that there is no another presentation $\mathcal{L} \simeq \mathcal{K}_X - h_d^0 + \zeta_e^o$.

(iii) Assume $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ with $d + e = \text{gon}(X)$ admits two other different presentations $\mathcal{L} \simeq \mathcal{K}_X - h_d^0 + \zeta_e^o \simeq \mathcal{K}_X - f_d^0 + \tau_e^o$. Due to Theorem 2.4, we have

the following:

$$\begin{aligned} g_d^0 + \zeta_e^o &\simeq h_d^0 + \xi_e^o, \quad g_d^0 + \zeta_e^o \neq h_d^0 + \xi_e^o, \\ g_d^0 + \tau_e^o &\simeq f_d^0 + \xi_e^o, \quad g_d^0 + \tau_e^o \neq f_d^0 + \xi_e^o, \end{aligned}$$

and both $|\mathcal{O}_X(g_d^0 + \zeta_e^o)|$ and $|\mathcal{O}_X(g_d^0 + \tau_e^o)|$ are pencils of degree $\text{gon}(X)$. If $|\mathcal{O}_X(g_d^0 + \zeta_e^o)| = |\mathcal{O}_X(g_d^0 + \tau_e^o)|$, then

$$g_d^0 + \zeta_e^o \simeq g_d^0 + \tau_e^o \quad \text{and} \quad h_d^0 + \xi_e^o \simeq f_d^0 + \xi_e^o,$$

whence $\zeta_e^o = \tau_e^o$ and $h_d^0 = f_d^0$ since both of d and e are smaller than $\text{gon}(X)$ by the conditions that $d + e = \text{gon}(X)$, $g_d^0 > 0$ and $\xi_e^o > 0$. It is a contradiction to the assumption that $\mathcal{L} \simeq \mathcal{K}_X - h_d^0 + \zeta_e^o$ and $\mathcal{L} \simeq \mathcal{K}_X - f_d^0 + \tau_e^o$ are distinct. Hence two pencils $|\mathcal{O}_X(g_d^0 + \zeta_e^o)|$ and $|\mathcal{O}_X(g_d^0 + \tau_e^o)|$ are different. This gives the result (iii). Thus we complete the proof of the theorem. \square

Corollary 2.9. *Let X be an n -gonal curve. For $0 < e < n$, choose two distinct divisors $g_{n-e}^0 + \zeta_e^o$, $h_{n-e}^0 + \xi_e^o \in g_n^1$ with $(g_{n-e}^0, \xi_e^o) = 0$ and $(h_{n-e}^0, \zeta_e^o) = 0$. Then we have a nonspecial line bundle \mathcal{L} with $\text{Ova}(\mathcal{L}) = e - 2$ which is distinctly presented by $\mathcal{K}_X - h_{n-e}^0 + \zeta_e^o$ and $\mathcal{K}_X - g_{n-e}^0 + \xi_e^o$. Moreover, if X has a unique g_n^1 , then $\mathcal{L} \simeq \mathcal{K}_X - h_{n-e}^0 + \zeta_e^o$ is the only different minimal presentations from $\mathcal{L} \simeq \mathcal{K}_X - g_{n-e}^0 + \xi_e^o$.*

Remark 2.10. (i) *Whenever we take arbitrary g_d^0 and ξ_e^o on X with $d + e \leq \text{gon}(X)$ and $(g_d^0, \xi_e^o) = 0$, we obtain a line bundle $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^o$ which is in itself a minimal presentation, equivalently, $\text{Ova}(\mathcal{L}) = e - 2$. In particular, if $e \geq 3$ (\mathcal{L} is very ample) and $d + e < \text{gon}(X)$, then $\langle \xi_e^o \rangle_{\mathcal{L}}$ is a unique e -secant $(e - 2)$ -plane and has no s -secant $(s - 2)$ -planes for any $s \leq e - 1$.*

(ii) *Let X be an n -gonal curve with a unique g_n^1 . For any number e with $0 < e < n$, X has infinitely many nonspecial line bundles \mathcal{L} satisfying $\deg \mathcal{L} = 2g - 2 - n + 2e$ and $\text{Ova}(\mathcal{L}) = e - 2$ by Corollary 2.9, since two different general divisors $g_{n-e}^0 + \zeta_e^o$, $h_{n-e}^0 + \xi_e^o \in g_n^1$ satisfy the conditions $(g_{n-e}^0, \xi_e^o) = 0$ and $(h_{n-e}^0, \zeta_e^o) = 0$. In the case $e \geq 3$, the embedded curve $\varphi_{\mathcal{L}}(X)$ has exactly two e -secant $(e - 2)$ -planes and has no s -secant $(s - 2)$ -planes for $s \leq e - 1$.*

(iii) *If $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^o$ with $d + e = \text{gon}(X)$, then Theorem 2.8 (iii) implies the following inequality:*

$$\begin{aligned} &\#\{ e\text{-secant } (e - 2)\text{-planes of } \varphi_{\mathcal{L}}(X) \} \\ &\leq 1 + \#\{ g_{\text{gon}(X)}^1 \mid h^0(X, g_{\text{gon}(X)}^1(-g_d^0)) \geq 1, \quad h^0(X, g_{\text{gon}(X)}^1(-\xi_e^o)) \geq 1 \}. \end{aligned}$$

We can see the exactness of the condition $d + e \leq \text{gon}(X)$ in Theorem 2.8 through the following example.

Example 2.11. Let X be a smooth curve. Choose two distinct general divisors $g_d^0 + \xi_{e-b}^0$, $g_{d-b}^0 + \xi_e^0 \in g_{\text{gon}(X)}^1$ with $b > 0$. Note that $d+e = \text{gon}(X) + b > \text{gon}(X)$ and an equivalence $\mathcal{K}_X - g_{d-b}^0 + \xi_{e-b}^0 \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$. The general choices imply that $(g_d^0, \xi_e^0) = 0$ and $(g_{d-b}^0, \xi_{e-b}^0) = 0$ and hence $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0 \simeq \mathcal{K}_X - g_{d-b}^0 + \xi_{e-b}^0$ are well defined presentations. This means that $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ is not minimal.

The following example is comparable to Remark 2.10, (iii).

Example 2.12. Let X be an m -fold covering of an elliptic curve Γ via $\phi : X \rightarrow \Gamma$. Assume that $\mathcal{L} \simeq \mathcal{K}_X - \phi^*(P) + \phi^*(Q)$ for two distinct points $P, Q \in \Gamma$. Then, for an arbitrary $R \in \Gamma$, there is a point $S \in \Gamma$ such that $\mathcal{L} \simeq \mathcal{K}_X - \phi^*(S) + \phi^*(R)$. Specifically, if $\text{gon}(X) = 2m$, then there are infinitely many $g_{\text{gon}(X)}^1$ on X and the line bundle \mathcal{L} has infinitely many minimal presentations of type (m, m) with $2m = \text{gon}(X)$.

Proof. Let R be an arbitrary point of Γ with $R \neq Q$. Since $h^0(\Gamma, \mathcal{O}_\Gamma(P+R)) = 2$, we can choose a point S of Γ such that $P + R \simeq Q + S$. Then we have

$$\phi^*(P) + \phi^*(R) \simeq \phi^*(Q) + \phi^*(S),$$

which gives

$$\mathcal{L} \simeq \mathcal{K}_X - \phi^*(P) + \phi^*(Q) \simeq \mathcal{K}_X - \phi^*(S) + \phi^*(R).$$

Thus the result follows. \square

In addition, observe the line bundles of degree $2g - 2$ with respect to secant properties, since it is interesting to distinguish properties of a nonspecial line bundle of that degree from the special one which is canonical.

Remark 2.13. Let \mathcal{L} be a very ample line bundle with $\deg \mathcal{L} = 2g - 2$.

(i) The special case: \mathcal{L} is the canonical line bundle \mathcal{K}_X on a nonhyperelliptic curve X , for which the embedded curve $\varphi_{\mathcal{K}_X}(X)$ has at least one-dimensional family of $\text{gon}(X)$ -secant $(\text{gon}(X) - 2)$ -planes but has no s -secant $(s - 2)$ -planes for $s \leq \text{gon}(X) - 1$.

(ii) The nonspecial case: Let e an arbitrary number with $3 \leq e \leq \frac{\text{gon}(X)}{2}$ (resp. $3 \leq e < \frac{\text{gon}(X)}{2}$). By Corollary 2.9, X carries very ample nonspecial line bundles

$$\mathcal{L} \simeq \mathcal{K}_X - g_e^0 + \xi_e^o,$$

for each of which $\varphi_{\mathcal{L}}(X)$ has an (resp. unique) e -secant $(e - 2)$ -plane but has no s -secant $(s - 2)$ -planes for any $s \leq e - 1$. Furthermore, if X is a general k -gonal, then the range of the number e can be extended up to $3 \leq e < \frac{g+1}{4}$ (see Corollary 3.4 and Remark 3.3).

(iii) Assume that X is an m -fold covering of an elliptic curve Γ with $\text{gon}(X) = 2m$. Let $\mathcal{L} \simeq \mathcal{K}_X - \phi^*(P) + \phi^*(Q)$ for two distinct points $P, Q \in \Gamma$. Example

2.12 implies that the curve $\varphi_{\mathcal{L}}(X)$ has no n -secant $(n-2)$ -planes for $n \leq m-1$ and has infinitely many m -secant $(m-2)$ -planes such that

$$\{ m\text{-secant } (m-2)\text{-planes} \} = \{ \langle \phi^*(R) \rangle_{\mathcal{L}} \mid R \in \Gamma \}.$$

Consequently, the curve $\varphi_{\mathcal{L}}(X)$ lies on an $(m-1)$ dimensional scroll S over Γ . Moreover, any two distinct m -secant $(m-2)$ -planes have no common points, since the Riemann-Roch Theorem gives $\dim \langle \phi^*(R_1 + R_2) \rangle_{\mathcal{L}} = 2m-3$ due to $\mathcal{L}(-R_1 - R_2) \simeq \mathcal{K}_X - \phi^*(P) - \phi^*(R_2)$ and $h^0(X, \mathcal{O}_X(\phi^*(P) + \phi^*(R_2))) = 2$. (iv) Assume that X is a double covering of a smooth curve Y of genus γ via $\phi : X \rightarrow Y$. For each $e \leq \frac{g-2\gamma}{2}$, there are nonspecial line bundles

$$\mathcal{L} \simeq \mathcal{K}_X - g_e^0 + \xi_e^0,$$

for which $\varphi_{\mathcal{L}}(X)$ has an e -secant $(e-2)$ -plane but has no s -secant $(s-2)$ -planes for any $s \leq e-1$ (see Corollary 3.5).

Note that one of the natural ways to construct nonspecial line bundles on a smooth curve X is to use a g_d^r , whose existence on X is already well known, such as $\mathcal{L} \simeq \mathcal{K}_X - g_d^r + \xi_e^0$ for some ξ_e^0 on X . In this case, we may raise a question concerning the minimality of $\mathcal{L} \simeq \mathcal{K}_X - g_{d-\beta}^0 + \xi_{e-\beta}^0$, where $g_{d-\beta}^0 := g_d^r - (\xi_e^0, \tilde{D})$ and $\xi_{e-\beta}^0 := \xi_e^0 - (\xi_e^0, \tilde{D})$ for some $\tilde{D} \in g_d^r$ with $\deg(\xi_e^0, \tilde{D}) = \beta$, $\beta =: \max\{\deg(\xi_e^0, D) \mid D \in g_d^r\}$. The following theorem gives sufficient conditions for the minimality.

Theorem 2.14. *Let X be a smooth curve with a complete g_d^r and let \mathcal{L} be a nonspecial line bundle on X given by $\mathcal{K}_X - g_d^r + \xi_e^0$ for some ξ_e^0 on X . Set $\beta := \max\{\deg(\xi_e^0, D) \mid D \in g_d^r\}$. Assume that $\dim |g_d^r + G| = r$ for any $G \geq 0$ with $\deg G \leq e - \beta - 1$. If $D \in g_d^r$ satisfies $\deg(\xi_e^0, D) = \beta$ then $\mathcal{L} \simeq \mathcal{K}_X - g_{d-\beta}^0 + \xi_{e-\beta}^0$ is a minimal presentation, where $g_{d-\beta}^0 := g_d^r - (\xi_e^0, D)$, $\xi_{e-\beta}^0 := \xi_e^0 - (\xi_e^0, D)$.*

Proof. Since \mathcal{L} is nonspecial, we obtain $\beta \leq e-1$, and $\mathcal{L} \simeq \mathcal{K}_X - g_{d-\beta}^0 + \xi_{e-\beta}^0$ is a minimal presentation in case $\beta = e-1$. Assume that for $\beta \leq e-2$ there is another presentation $\mathcal{L} \simeq \mathcal{K}_X - h_t^0 + \zeta_s^0$ with $s \leq e - \beta - 1$. This gives $\mathcal{K}_X - g_d^r + \xi_e^0 \simeq \mathcal{K}_X - h_t^0 + \zeta_s^0$, whence

$$|g_d^r + \zeta_s^0| = |h_t^0 + \xi_e^0|.$$

Since $\dim |g_d^r + G| = r$ for any $G \geq 0$ with $\deg G \leq e - \beta - 1$, we obtain

$$|h_t^0 + \xi_e^0| = |g_d^r + \zeta_s^0| = g_d^r + \zeta_s^0,$$

which means that $\zeta_s^0 \leq h_t^0 + \xi_e^0$ and thus $\zeta_s^0 \leq \xi_e^0$ due to $(h_t^0, \zeta_s^0) = 0$. From the equality $|h_t^0 + \xi_e^0| = g_d^r + \zeta_s^0$ we get $\xi_e^0 - \zeta_s^0 \leq F \in g_d^r$ and so $(\xi_e^0 - \zeta_s^0, F) \leq (\xi_e^0, F)$, which is contrary to the definition of β since $e - s \geq \beta + 1$ for $s \leq e - \beta - 1$. Thus $\mathcal{L} \simeq \mathcal{K}_X - g_{d-\beta}^0 + \xi_{e-\beta}^0$ is a minimal presentation. \square

On the other hand, there is an example such that the minimality fails when $\dim |g_d^r + G| > r$ for some $G \geq 0$ with $\deg G \leq e - \beta - 1$.

Example 2.15. Let X be a linearly normal smooth curve of type (a, b) with $a \geq b \geq 2$ on a smooth quadric surface in \mathbb{P}^3 . Choose a subdivisor ξ_e^0 of a general $H \in |\mathcal{O}_X(1)|$ with $e \geq a+1$. Let $\mathcal{L} \simeq \mathcal{K}_X - g_a^1 + \xi_e^0$. Then, $\max\{\deg(\xi_e^0, D) \mid D \in g_a^1\}$ is equal to one. However the presentation $\mathcal{L} \simeq \mathcal{K}_X - g_{a-1}^0 + \xi_{e-1}^0$ with $\xi_{e-1}^0 := \xi_e^0 - P \geq 0$ and $g_{a-1}^0 := g_a^1 - P$ is not minimal.

Proof. The general choice of H implies that $\max\{\deg(\xi_e^0, D) \mid D \in g_a^1\} = 1$ and $h^0(X, \mathcal{O}_X(H - \xi_e^0)) = 1$. Take a $Q \in X$ with $Q \leq (H - \xi_e^0)$. Set $h_{a+b-e-1}^0 := H - \xi_e^0 - Q$, $\zeta_{b-1}^0 := g_b^1 - Q$. From the equivalence $(H - \xi_e^0 - Q) + \xi_e^0 \simeq g_a^1 + (g_b^1 - Q)$ we obtain

$$\mathcal{K}_X - g_a^1 + \xi_e^0 \simeq \mathcal{K}_X - h_{a+b-e-1}^0 + \zeta_{b-1}^0.$$

This means that $\mathcal{L} \simeq \mathcal{K}_X - g_{a-1}^0 + \xi_{e-1}^0$ is not a minimal presentation for $b \leq a \leq e - 1$. \square

Note that this example does not satisfy the hypothesis of Theorem 2.14, since $\dim |g_a^1 + G| = \dim |H - Q| = 2$ for $G := g_b^1 - Q$ of degree $b - 1 \leq e - 1 - \beta$.

3. Minimal presentations on multiple coverings

In this section, we investigate sufficient conditions for the minimality of presentations of nonspecial line bundles on multiple coverings.

Since every trivial presentation is minimal, we consider only nontrivial presentations ($g_d^0 \neq 0$) and so we use a notation ξ_e^0 instead of E due to Proposition 2.2, (v). Thus the aim of this section is to explore sufficient conditions for the minimality of nontrivial presentations such as $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ on multiple coverings. We also assume $e \geq 2$ which is necessary for the line bundle $\mathcal{K}_X - g_d^0 + \xi_e^0$ to be globally generated.

First, we examine necessary conditions for $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ to be minimal on curves X with $\phi : X \rightarrow \mathbb{P}^1$ which are the simplest coverings to deal with.

Proposition 3.1. Let X admit an n -fold covering $\phi : X \rightarrow \mathbb{P}^1$ and let \mathcal{L} be a nonspecial line bundle on X . If $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ is a minimal presentation, then $\deg(g_d^0, \phi^*(P)) + \deg(\xi_e^0, \phi^*(Q)) \leq n$ for any $P, Q \in \mathbb{P}^1$.

Proof. Let $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ is a minimal presentation. Suppose that $(g_d^0, \phi^*(P)) := D > 0$ and $(\xi_e^0, \phi^*(Q)) := E > 0$ for $P, Q \in \mathbb{P}^1$. The equivalence $\phi^*(P) \simeq \phi^*(Q)$ gives

$$\mathcal{K}_X - g_d^0 + \xi_e^0 \simeq \mathcal{K}_X - g_d^0 + D - (\phi^*(Q) - E) + \xi_e^0 - E + (\phi^*(P) - D).$$

If we set $E' := \phi^*(P) - D$, $D' := \phi^*(Q) - E$, then we have

$$\mathcal{L} \simeq \mathcal{K}_X - (g_d^0 - D + D') + (\xi_e^0 - E + E'),$$

with $g_d^0 - D + D' \geq 0$ and $\xi_e^0 - E + E' \geq 0$. Take a divisor $B \geq 0$ such that $|g_d^0 - D + D' - B| =: g_t^0$ and $|\xi_e^0 - E + E' - B| =: \xi_s^0$ with $(g_t^0, \xi_s^0) = 0$. This gives a presentation

$$\mathcal{L} \simeq \mathcal{K}_X - g_t^0 + \xi_s^0 \quad \text{with } s + t \leq d + e - \deg(D + E) + \deg(D' + E'),$$

whence the minimality of $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ implies $\deg(D + E) \leq \deg(D' + E')$. This yields $\deg(D + E) \leq n$ since $\phi^*(P + Q) = D + D' + E + E'$. Accordingly the theorem is verified. \square

In fact, the conclusion that $\deg(g_d^0, \phi^*(P)) + \deg(\xi_e^0, \phi^*(Q)) \leq n$ for any P, Q of the base curve also becomes a sufficient condition for the minimality of $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ on multiple coverings in some restricted range of $d + e$ as follows.

Theorem 3.2. *Assume that X admits an n -fold covering morphism $\phi : X \rightarrow Y$ for smooth curve Y . Choose g_d^0 and ξ_e^0 on X with $(g_d^0, \xi_e^0) = 0$, $e \geq 2$ and $d + e \leq \mu$, where $\mu := \min\{\deg \mathcal{N} \mid \mathcal{N} : \text{globally generated and not composed with } \phi\}$. If $\deg(g_d^0, \phi^*(P)) + \deg(\xi_e^0, \phi^*(Q)) \leq n$ for any $P, Q \in Y$, then we have a nonspecial line bundle $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ which is in itself a minimal presentation.*

Proof. Suppose that $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ has another presentation $\mathcal{L} \simeq \mathcal{K}_X - h_t^0 + \zeta_s^0$ with $s \leq e - 1$. The condition $s \leq e - 1$ also implies $t \leq d - 1$ since $d - e = t - s$. According to Theorem 2.4 we have

$$g_d^0 + \zeta_s^0 \simeq h_t^0 + \xi_e^0, \quad g_d^0 + \zeta_s^0 \neq h_t^0 + \xi_e^0, \quad h^0(X, \mathcal{O}_X(g_d^0 + \zeta_s^0)) \geq 2,$$

whence

$$|g_d^0 + \zeta_s^0| = |g_t^0 + \xi_e^0| = \phi^*(g_m^r) + B, \quad r \geq 1, \quad B : \text{base locus},$$

since $d + s < d + e \leq \mu$. Then we have the following decompositions:

$$(1) \quad \begin{aligned} g_d^0 &= \Lambda + B_1, \quad h_t^0 = \Lambda' + B'_1, \\ \zeta_s^0 &= \Sigma' + B'_2, \quad \xi_e^0 = \Sigma + B_2, \end{aligned}$$

such that

$$\begin{aligned} |\Lambda + \Sigma'| &= |\Lambda' + \Sigma| = \phi^*(g_m^r), \\ B &= B_1 + B'_2 = B'_1 + B_2, \end{aligned}$$

for some effective divisors $\Lambda, \Lambda', \Sigma, \Sigma', B_k, B'_k$ $k = 1, 2$. Thus there are points $P_i, Q_j \in Y$, $i, j = 1, \dots, m$ such that

$$\Lambda + \Sigma' = \phi^*(P_1 + \dots + P_m) \quad \text{and} \quad \Lambda' + \Sigma = \phi^*(Q_1 + \dots + Q_m).$$

Accordingly, for each $i, j \in \{1, \dots, m\}$ we can set

$$\begin{aligned} D_i + E'_i &= \phi^*(P_i) \quad \text{for some } D_i \leq \Lambda, \quad E'_i \leq \Sigma', \\ D'_j + E_j &= \phi^*(Q_j) \quad \text{for some } D'_j \leq \Lambda', \quad E_j \leq \Sigma. \end{aligned}$$

Thus the hypothesis on $g_d^0 + \xi_e^0$ in the theorem gives

$$\deg(D_i + E_j) \leq \deg(g_d^0, \phi^*(P_i)) + \deg(\xi_e^0, \phi^*(Q_j)) \leq n,$$

whence

$$\deg(D'_j + E'_i) \geq n \geq \deg(D_i + E_j)$$

due to $D_i + E_j + D'_j + E'_i = \phi^*(P_i + Q_j)$. This implies

$$(2) \quad \deg(\Lambda' + \Sigma') \geq \deg(\Lambda + \Sigma),$$

since $\Lambda + \Sigma = \sum_{i=1}^n D_i + \sum_{j=1}^n E_j$ and $\Lambda' + \Sigma' = \sum_{j=1}^n D'_j + \sum_{i=1}^n E'_i$.

On the other hand, because B is a base locus, we have $B = B_1 + B'_2 = B'_1 + B_2$ as divisors, whence

$$B'_1 = B_1, \quad B'_2 = B_2 \quad \text{as divisors}$$

by the conditions $(g_d^0, \xi_e^0) = 0$ and $(h_t^0, \zeta_s^0) = 0$. Accordingly, by (1) we obtain

$$\deg \Lambda = d - \deg B_1 \geq 1 + t - \deg B'_1 = 1 + \deg \Lambda'$$

$$\deg \Sigma = e - \deg B_2 \geq 1 + s - \deg B'_2 = 1 + \deg \Sigma'$$

for $d \geq t + 1$ and $e \geq s + 1$. This gives that

$$\deg(\Lambda + \Sigma) \geq 2 + \deg(\Lambda' + \Sigma'),$$

which is contrary to (2). Thus $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ is a minimal presentation. \square

Remark 3.3. Whenever we take general g_d^0 and ξ_e^0 on a multiple covering X with $d + e \leq \mu$ and $(g_d^0, \xi_e^0) = 0$, we obtain a nonspecial line bundle $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ which is in itself a minimal presentation since the general choices of g_d^0 and ξ_e^0 imply $\deg(g_d^0, \phi^*(P)) + \deg(\xi_e^0, \phi^*(Q)) \leq n$ for any $P, Q \in Y$.

Corollary 3.4. Let X be a general n -gonal curve of genus $g \geq 4$ via $\phi : X \rightarrow \mathbb{P}^1$. Choose g_d^0 and ξ_e^0 on X with $(g_d^0, \xi_e^0) = 0$ and $d + e \leq \frac{g+3}{2}$. Then $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ is in itself a minimal presentation if and only if $\deg(g_d^0, \phi^*(P)) + \deg(\xi_e^0, \phi^*(Q)) \leq n$ for any $P, Q \in \mathbb{P}^1$.

Proof. Assume $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ with $\deg(g_d^0, \phi^*(P)) + \deg(\xi_e^0, \phi^*(Q)) \leq n$ for any $P, Q \in \mathbb{P}^1$. According to Theorem (2.6) in [2], a general n -gonal curve X has a unique g_n^1 and any globally generated line bundle \mathcal{M} on X with $\deg \mathcal{M} \leq \frac{g+1}{2}$ is composed with the n -fold covering morphism associated to the g_n^1 . Hence Theorem 3.2 implies that $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ is a minimal presentation for $d + e \leq \frac{g+3}{2}$. The converse trivially comes from Proposition 3.1. \square

Specifically, we obtain the following for a simple covering $\phi : X \rightarrow Y$, since $\mu \geq \frac{g(X) - ng(Y)}{n-1} + 1$ by the Castelnuovo-Severi inequality.

Corollary 3.5. *Let a smooth curve X genus $g \geq 2$ admit a simple n -fold covering morphism $\phi : X \rightarrow Y$ for a smooth curve Y of genus γ . If g_d^0 and ξ_e^0 on X with $d+e \leq [\frac{g-n\gamma}{n-1}] + 1$ satisfy that $(g_d^0, \xi_e^0) = 0$ and $\deg(g_d^0, \phi^*(P)) + \deg(\xi_e^0, \phi^*(Q)) \leq n$ for any $P, Q \in Y$, then we obtain a nonspecial line bundle $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ which is in itself a minimal presentation. Specifically, for a double covering case the presentation $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_e^0$ with $d+e \leq g-2\gamma+1$ is minimal if $\deg(g_d^0, \phi^*(Q)) \leq 1$ and $\deg(\xi_e^0, \phi^*(Q)) \leq 1$ for any $Q \in Y$.*

Corollary 3.6. *Let X be an n -fold covering of \mathbb{P}^1 via $\phi : X \rightarrow \mathbb{P}^1$ and μ be the same as in Theorem 3.2. Assume that ξ_{e+r}^0 satisfies $\deg(\xi_{e+r}^0, \phi^*(Q)) \leq 1$ and $\phi(P_1) \neq \phi(P_2)$ for any $Q \in \mathbb{P}^1$ and $P_1 + P_2 \leq \xi_{e+r}^0$. If $\max\{rn, rn - r + e\} < \mu$ then $\mathcal{L} \simeq \mathcal{K}_X - rg_n^1 + \xi_{e+r}^0$ is a nonspecial line bundle minimally presented by $\mathcal{L} \simeq \mathcal{K}_X - g_{rn-r}^0 + \xi_e^0$, where $g_{rn-r}^0 := rg_n^1(-\xi_r^0)$, $\xi_e^0 := \xi_{e+r}^0 - \xi_r^0$ for some $\xi_r^0 \leq \xi_{e+r}^0$.*

Proof. Note that $|rg_n^1| = g_{rn}^r$ for $rn < \mu$ and thus $h^0(X, \mathcal{O}_X(rg_n^1(-\xi_r^0))) = 1$ for $\xi_r^0 \leq \xi_{e+r}^0$ due to the condition that $\deg(\xi_{e+r}^0, \phi^*(Q)) \leq 1$ for any $Q \in \mathbb{P}^1$. Thus we can set $g_{rn-r}^0 := rg_n^1(-\xi_r^0)$, which satisfies that $(g_{rn-r}^0, \xi_e^0) = 0$ for $\xi_e^0 := \xi_{e+r}^0 - \xi_r^0$. Accordingly, $\mathcal{L} \simeq \mathcal{K}_X - rg_n^1 + \xi_{e+r}^0$ admits a well defined presentation $\mathcal{L} \simeq \mathcal{K}_X - g_{rn-r}^0 + \xi_e^0$. Its minimality comes from Theorem 3.2, since $\deg(g_{rn-r}^0, \phi^*(Q)) \leq n-1$, $\deg(\xi_e^0, \phi^*(Q)) \leq 1$ and $\phi(P_1) \neq \phi(P_2)$ for any $Q \in \mathbb{P}^1$ and $P_1 + P_2 \leq \xi_{e+r}^0$. \square

Remark 3.7. *Let X be an n -gonal curve of genus g via $\mu : X \rightarrow \mathbb{P}^1$ and μ be the same as in Theorem 3.2. Assume that $\max\{rn, rn - kr + e\} < \mu$. Choose a ξ_{e+kr}^0 with $k \geq 1$ such that $\deg(\xi_{e+kr}^0, \phi^*(Q_i)) = k$ for distinct $Q_1, \dots, Q_r \in \mathbb{P}^1$ and $\deg(\xi_{e+kr}^0, \phi^*(Q)) \leq k$ for any $Q \neq Q_i, i = 1, \dots, r$. Let $\xi_{kr}^0 := \sum_{i=1}^r (\xi_{e+kr}^0, \phi^*(Q_i))$. Then, we have $h^0(X, \mathcal{O}_X(rg_n^1(-\xi_{kr}^0))) = 1$ since $|rg_n^1| = g_{nr}^r = |\phi^*(\sum_{i=1}^r Q_i)|$. Set $g_{rn-kr}^0 := rg_n^1(-\xi_{kr}^0)$ and $\xi_e^0 := \xi_{e+kr}^0 - \xi_{kr}^0$. Then $\mathcal{L} \simeq \mathcal{K}_X - rg_n^1 + \xi_{e+kr}^0$ is a nonspecial line bundle minimally presented by $\mathcal{L} \simeq \mathcal{K}_X - g_{rn-kr}^0 + \xi_e^0$ due to Theorem 3.2. The proof is very similar to Corollary 3.6.*

Now, we consider a minimal presentation problem for nonspecial line bundles on X with a simple morphism $\phi : X \rightarrow Y$ for a smooth plane curve Y , since it is possible to use some theories on linear systems on smooth plane curves. Here, ϕ is said to be simple if it does not factor through. In §4, the result on this will be applied to investigate property (N_p) of line bundles on such curves.

Theorem 3.8 ([14], p.82). *Let C be a smooth plane curve of degree d . Let g_n^1 be a linear system on C . And let \mathbb{P}_k be the projective space parameterizing effective divisors of degree k on \mathbb{P}^2 . If $g_n^1 = \mathbb{P}.C - F(\mathbb{P}.C)$ for a pencil \mathbb{P} of \mathbb{P}_k , then $n \geq k(d-k)$, where $F(\mathbb{P}.C) := \cap\{E|E \in \mathbb{P}.C\}$.*

This theorem gives the following lemma.

Lemma 3.9. *Let C be a smooth plane curve of degree d . If \mathcal{D} is a base point free complete linear system on C with $\dim \mathcal{D} \geq 1$ and $\deg \mathcal{D} \leq 2d - 5$, then \mathcal{D} is equal to either g_{d-1}^1 or g_d^2 .*

From Lemma 3.9 we obtain conditions for the minimality of presentations of typical nonspecial line bundles such as $\mathcal{K}_X - \phi^* g_d^2 + \xi_{e+2}$ and $\mathcal{K}_X - \phi^* g_{d-1}^1 + \xi_{e+1}$ on a multiple covering X of a smooth plane curve Y of degree d via $\phi : X \rightarrow Y$.

Theorem 3.10. *Let a smooth curve X of genus g admit a morphism $\phi : X \rightarrow Y \subset \mathbb{P}^2$ which does not factor through for a smooth plane curve Y of degree $d \geq 5$ with $g > ng(Y) + n(n-1)d$ for $n := \deg \phi \geq 2$. Let $\delta_\epsilon := \min\{\lfloor \frac{g-ng(Y)}{n-1} \rfloor - nd + 3, nd - 5n + 3\} + \epsilon(n-1)$ in case $n \geq 2$; $\delta_\epsilon := nd - 5n + 3$ in case $n = 1$; $\epsilon = 0, 1$. (i) If we take a ξ_e^0 with $e \leq \delta_0$ such that $\deg(\xi_e^0, \phi^*(H)) \leq 2$ for any $H \in g_d^2$, then the line bundle $\mathcal{L} \simeq \mathcal{K}_X - \phi^*(g_d^2) + \xi_e^0$ carries a natural minimal presentation $\mathcal{L} \simeq \mathcal{K}_X - g_{nd-2}^0 + \xi_{e-2}^0$, where $g_{nd-2}^0 := \phi^*(g_d^2) - (P_1 + P_2)$, $\xi_{e-2}^0 := \xi_e^0 - (P_1 + P_2) \geq 0$. (ii) If we take a ξ_e^0 with $e \leq \delta_1$ satisfying that $\deg(\xi_e^0, \phi^*(H - \tilde{Q})) \leq 1$ for any $H \in g_d^2$ with $H \geq \tilde{Q}$ and $\deg(\xi_e^0, \phi^*(H)) \leq n + 1$ for any $H \in g_d^2$, then $\mathcal{L} \simeq \mathcal{K}_X - \phi^*(g_d^2(-\tilde{Q})) + \xi_e^0$ carries a natural minimal presentation $\mathcal{L} \simeq \mathcal{K}_X - g_{nd-n-1}^0 + \xi_{e-1}^0$, where $g_{nd-n-1}^0 := \phi^*(g_d^2(-\tilde{Q})) - P$, $\xi_{e-1}^0 := \xi_e^0 - P \geq 0$, $P \in X$.*

Proof. First, we verify the theorem in the case $n \geq 2$, since the theorem for $n = 1$ can be shown easily through a similar proof.

(i) Note that we get $\beta = 2$ due to $\deg(\xi_e^0, \phi^*(H)) \leq 2$ for any $H \in g_d^2$, where $\beta := \max\{\deg(\xi_e^0, D) \mid D \in g_d^2\}$. By Theorem 2.14, it suffices to show that $\dim |\phi^*(g_d^2) + G| = 2$ for any $G \geq 0$ with $\deg G \leq e - 3$. Since $\deg(\phi^*(g_d^2) + G) \leq nd + e - 3 \leq \min\{\lfloor \frac{g-ng(Y)}{n-1} \rfloor, n(2d-5)\}$, the Castelnuovo-Severi inequality implies that $|\phi^*(g_d^2) + G|$ is composed with ϕ and so Lemma 3.9 gives $|\phi^*(g_d^2) + G| = \phi^*(g_d^2) + G$, that is, $\dim |\phi^*(g_d^2) + G| = 2$. This completes the proof of (i).

(ii) According to the condition that $\deg(\xi_e^0, \phi^*(H - \tilde{Q})) \leq 1$ for any $H \in g_d^2$ with $H \geq \tilde{Q}$, the number $\beta := \max\{\deg(\xi_e^0, D) \mid D \in \phi^*(g_d^2(-\tilde{Q}))\}$ is equal to one and thus \mathcal{L} admits a presentation $\mathcal{L} \simeq \mathcal{K}_X - g_{nd-n-1}^0 + \xi_{e-1}^0$, where $g_{nd-n-1}^0 := \phi^*(g_d^2(-\tilde{Q})) - P$, $\xi_{e-1}^0 := \xi_e^0 - P \geq 0$, $P \in X$. Assume that there is another presentation

$$\mathcal{L} \simeq \mathcal{K}_X - \phi^*(g_d^2(-\tilde{Q})) + \xi_e^0 \simeq \mathcal{K}_X - h_t^0 + \zeta_s^0$$

with $s \leq e - 2$ which also means $t \leq nd - n - 2$. This yields that

$$|\phi^*(g_d^2(-\tilde{Q})) + \zeta_s^0| = |h_t^0 + \xi_e^0| =: g_{nd-n+s}^\alpha.$$

Note that $e \leq \delta_1$ gives that $nd - n + s \leq nd - n + e - 2 \leq \min\{\lfloor \frac{g-ng(Y)}{n-1} \rfloor, n(2d-5)\}$. Accordingly, by the Castelnuovo-Severi inequality the linear system g_{nd-n+s}^α is

composed with ϕ , whence by Lemma 3.9 we conclude

$$g_{nd-n+s}^\alpha = \phi^* g_d^2 + B_{s-n \geq 0} \quad \text{or} \quad \phi^* g_d^2(-\tilde{Q}) + B_s,$$

B_{s-n} , B_s : base loci. Here, we trivially obtain that $B_{s-n} \leq \zeta_s^0$ and $B_s = \zeta_s^0$. Assume that $g_{nd-n+s}^\alpha = \phi^* g_d^2 + B_{s-n \geq 0}$. Then there is a $H \in g_d^2$ such that $\deg(\xi_e^0, \phi^* H) \geq nd-t \geq n+2$ due to $|h_t^0 + \xi_e^0| = \phi^* g_d^2 + B_{s-n}$ and $t \leq nd-n-2$. This is absurd since $\deg(\xi_e^0, \phi^*(H)) \leq n+1$ for any $H \in g_d^2$. It forces that

$$g_{nd-n+s}^\alpha = \phi^* g_d^2(-\tilde{Q}) + B_s.$$

This means that $\deg(\xi_e^0, \phi^*(H-\tilde{Q})) \geq nd-n-t \geq 2$ for some $H \in g_d^2$ with $\tilde{Q} \leq H$, since $h_t^0 + \xi_e^0 \in \phi^* g_d^2(-\tilde{Q}) + B_s$. Accordingly, we also meet a contradiction to $\deg(\xi_e^0, \phi^*(H-\tilde{Q})) \leq 1$ for any $H \in g_d^2$ with $H \geq \tilde{Q}$. As a result, the presentation $\mathcal{L} \simeq \mathcal{K}_X - g_{nd-n-1}^0 + \xi_{e-1}^0$ is minimal. This completes the proof of the theorem for $n \geq 2$.

Next, consider the case of $n = 1$ which means the biregularity of ϕ since Y is nonsingular. Thus any linear system on X is composed with ϕ and hence we can verify the theorem by using Lemma 3.9 and substituting $n = 1$ in the proof of the case $n \geq 2$. Finally, we obtain the result. \square

Note that the Riemann-Hurwitz Formula implies $g \geq ng(Y) - n + 1$ and thus the hypothesis $g > ng(Y) + n(n-1)d$ of Theorem 3.10 is not strong in case $d > n$.

4. Applications to Green-Lazarsfeld's conjecture on syzygies of curves

Consider a very ample line bundle \mathcal{L} on a smooth curve X and the homogeneous coordinate ring $S := \text{Sym}(H^0(X, \mathcal{L}))$ of $\varphi_{\mathcal{L}}(X)$ in $\mathbb{P}^r := \mathbb{P}H^0(X, \mathcal{L})$. Then we have a minimal free resolution of $S(X)$ as a graded S -module as follows:

$$0 \rightarrow E_{r-1} \rightarrow \cdots \rightarrow E_p \rightarrow E_{p-1} \rightarrow \cdots \rightarrow E_1 \rightarrow S \rightarrow S(X) \rightarrow 0.$$

M. Green and R. Lazarsfeld have defined property (N_p) for \mathcal{L} , which means $E_0 = S$ and $E_i = \bigoplus^{\beta_{i,1}} S(-i-1)$ for all $1 \leq i \leq p$ ([7], Section 3). In their paper, they demonstrated that property (N_p) is closely related to the Clifford index $\text{Cliff}(X)$ of X which is an important birational numerical invariant of a smooth curve.

They verified in [7] that property (N_0) (normal generation) holds for any very ample line bundle \mathcal{L} on X with $\deg \mathcal{L} \geq 2g+1 - \text{Cliff}(X) - 2h^1(X, \mathcal{L})$. The exactness of this bound has shown in [7], [13], [5]: there are very ample line bundles with $\deg \mathcal{L} = 2g - \text{Cliff}(X) - 2h^1(X, \mathcal{L})$ which fail to be normally generated. In [8], they also proved that a line bundle \mathcal{L} of degree $2g+p$ on a nonhyperelliptic curve X satisfies (N_p) if and only if $\varphi_{\mathcal{L}}(X)$ has no $(p+2)$ -secant p -planes. In this context, M. Green and R. Lazarsfeld raised in [7] the following conjecture:

Conjecture 4.1 (Green-Lazarsfeld's conjecture on (N_p)). *Let \mathcal{L} be a very ample line bundle on a smooth curve X of genus g with $\deg \mathcal{L} \geq 2g + 1 + p - 2h^1(X, \mathcal{L}) - \text{Cliff}(X)$. If $\varphi_{\mathcal{L}}(X)$ has no $(p + 2)$ -secant p -planes then property (N_p) holds for \mathcal{L} .*

Remark 4.2. *Any spececial very ample line bundles on a general k -gonal curve X of genus g with $3 \leq k < [\frac{g}{2}] + 2$ satisfy Green-Lazarsfeld's conjecture on (N_p) by Theorem 2 in [3] and Theorem 1 in [6]. In fact, Theorem 2 in [3] implies that a general k -gonal curve X with $3 \leq k < [\frac{g}{2}] + 2$ satisfies Green's Conjecture: $K_{p,1}(X, K_X) = 0$ if and only if $p \geq g - \text{Cliff}(X) - 1$; and Theorem 1 in [6] says that if a very ample line bundle \mathcal{L} on X satisfies property (N_p) then $\mathcal{L}(-Q)$ has property (N_{p-1}) for any $Q \in X$.*

Now, note that minimal presentations of nonspecial line bundles give not only information on the existence of $(p + 2)$ -secant p -planes but also the construction of nonspecial line bundles with/without a $(p + 2)$ -secant p -plane. Accordingly, a minimal presentation can be an effective tool to observe the exactness of Conjecture 4.1.

Definition 4.3. *Let X be a non-hyperelliptic curve of genus g . A very ample line bundle \mathcal{L} on X with $\deg \mathcal{L} = 2g + p - 2h^1(X, \mathcal{L}) - \text{Cliff}(X)$ is called an extremal line bundle for Green-Lazarsfeld's conjecture on (N_p) if \mathcal{L} does not satisfy property (N_p) and $\varphi_{\mathcal{L}}(X)$ has no $(p + 2)$ -secant p -planes. Specifically, for $p = 0$ it was already defined by an extremal line bundle in [7].*

We will demonstrate that general n -gonal curves and some simple n -fold coverings of smooth plane curves carry nonspecial extremal line bundles for Green-Lazarsfeld's conjecture on (N_p) . Furthermore, the results also show how to construct extremal line bundles for Green-Lazarsfeld's conjecture on (N_p) on such curves.

Before going to our main results of this section, we consider that any line bundle \mathcal{L} of degree $2g + p (= 2g + p - 2h^1(X, \mathcal{L}) - \text{Cliff}(X))$ on a hyperelliptic curve X does not satisfy property (N_p) , whereas a line bundle \mathcal{L} of that degree on a nonhyperelliptic curve does not satisfy property (N_p) if and only if \mathcal{L} embeds X with a $(p + 2)$ -secant p -plane (see [8], Theorem 2). The following proposition explicitly shows that property (N_p) for the line bundle \mathcal{L} on hyperelliptic curves is regardless of the existence of such secant planes.

Proposition 4.4. *Let X be a hyperelliptic curve of genus $g \geq 2$. Choose two divisors $g_{d \geq 1}^0$ and ξ_{d+p+2}^0 on X with $(g_d^0, \xi_{d+p+2}^0) = 0$ for $p \geq 0$, $p + 2d \leq g - 1$. Then the nonspecial line bundle $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_{d+p+2}^0$ embeds X without a $(p + d + 1)$ -secant $(p + d - 1)$ -plane and does not satisfy property (N_p) .*

Proof. By Theorem 2 in [8] the line bundle \mathcal{L} does not satisfy the property (N_p) . Corollary 3.5 implies that $\mathcal{L} \simeq \mathcal{K}_X - g_d^0 + \xi_{p+d+2}^0$ is a minimal presentation for

$p+2d \leq g-1$, and hence \mathcal{L} is very ample and embeds X with no $(p+d+1)$ -secant $(p+d-1)$ -planes. \square

The following theorem shows that a general n -gonal curve X of genus g with $3 \leq n \leq [\frac{g-2}{2}]$ carries numerous nonspecial extremal line bundles \mathcal{L} for Green-Lazarsfeld's conjecture on (N_p) , that is, (1) $\deg \mathcal{L} = 2g + p - \text{Cliff}(X)$, (2) \mathcal{L} is very ample and does not satisfy property (N_p) , (3) $\varphi_{\mathcal{L}}(X)$ has no $(p+2)$ -secant p -planes.

Theorem 4.5. *Let X be a general n -gonal curve of genus g with $3 \leq n \leq [\frac{g-2}{2}]$. For $p \leq \frac{g-1}{2} - n$, the curve X carries nonspecial extremal line bundles for Green-Lazarsfeld's conjecture on (N_p) which are given by $\mathcal{L} \simeq \mathcal{K}_X - g_n^1 + \xi_{p+4}^0$ for some ξ_{p+4}^0 with $\deg(\xi_{p+4}^0, F) \leq 1$ for any $F \in g_n^1$.*

Proof. First, consider the case $p = 0$. By Corollary 3.4, a line bundle $\mathcal{L} \simeq \mathcal{K}_X - g_n^1 + \xi_4^0$ is minimally presented by

$$\mathcal{L} \simeq \mathcal{K}_X - g_{n-1}^0 + \xi_3^0, \quad g_{n-1}^0 := g_n^1(-P), \quad \xi_3^0 := \xi_4^0 - P, \quad P \in \xi_4^0,$$

due to $\deg(\xi_4^0, F) \leq 1$ for any $F \in g_n^1$. Thus \mathcal{L} is very ample. By the same arguments of (2.1) Theorem in [7], \mathcal{L} fails to be normally generated since $D := \xi_3^0 + P$ spans a line via the embedding $\varphi_{\mathcal{L}}$.

Next, consider $\mathcal{L} \simeq \mathcal{K}_X - g_n^1 + \xi_{p+4}^0$ with $0 < p \leq \frac{g-1}{2} - n$. Then \mathcal{L} admits a well defined presentation

$$\mathcal{L} \simeq \mathcal{K}_X - g_{n-1}^0 + \xi_{p+3}^0, \quad g_{n-1}^0 := g_n^1(-P), \quad \xi_{p+3}^0 := \xi_{p+4}^0 - P, \quad P \in \xi_{p+4}^0$$

since we have $(g_{n-1}^0, \xi_{p+3}^0) = 0$ by the condition $\deg(\xi_{p+4}^0, F) \leq 1$ for any $F \in g_n^1$. Corollary 3.4 implies the minimality of the presentation, since $p \leq \frac{g-1}{2} - n$, $\deg(\xi_{p+4}^0, F) \leq 1$ for any $F \in g_n^1$. Consequently, \mathcal{L} is very ample and embeds X with no $(p+2)$ -secant p -planes. Note that we have

$$\deg \mathcal{L} = 2g + p - \text{Cliff}(X),$$

since the Clifford index of a general n -gonal curve is equal to $n-2$ (see [4], [11]).

Suppose that \mathcal{L} satisfies property (N_p) . According to Theorem 1 in [6] (see Remark 4.2), the line bundle $\mathcal{M} \simeq \mathcal{K}_X - g_{n-1}^0 + \xi_3^0$ is normally generated, which cannot occur. Thus \mathcal{L} does not satisfy property (N_p) . As a consequence, \mathcal{L} is an extremal line bundle for Green-Lazarsfeld's conjecture on (N_p) . This completes the proof of the theorem. \square

In addition, we also want to show the existence of a nonspecial extremal line bundle for Green-Lazarsfeld's conjecture on (N_p) on a simple multiple covering of a smooth plane curve. To do this, we have to calculate the Clifford index of such curves. Accordingly, we examine the Clifford index of line bundles on multiple coverings. In the following, a line bundle \mathcal{M} on a multiple covering $\phi : X \rightarrow Y$

is said to be composed with ϕ if $\varphi_{\mathcal{M}}$ factors through ϕ , equivalently, $\mathcal{M} \simeq \phi^* \mathcal{N}$ and $h^0(X, \mathcal{M}) = h^0(Y, \mathcal{N})$.

Lemma 4.6. *Assume that a smooth curve X of genus g admits a simple n -fold covering morphism $\phi : X \rightarrow Y$ for a smooth curve Y of genus γ with $g > n\gamma$. Let \mathcal{M} be a globally generated line bundle on X with $\deg \mathcal{M} \leq g-1$ and $h^0(X, \mathcal{M}) \geq 2$. Then \mathcal{M} is composed with the morphism ϕ if*

$$\text{Cliff}(\mathcal{M}) < \min\left\{ \frac{g-n\gamma}{n-1} - 3, \frac{2(2n+\mu-3)}{(2n+\mu-1)^2}g-1 \right\},$$

where $\mu := \lceil \frac{2n(n-1)\gamma}{g-n\gamma} \rceil$.

Proof. By Lemma 5 in [9], if we show

$$\text{Cliff}(\mathcal{M}) < \min\left\{ \frac{g-n\gamma}{n-1} - 3, \frac{2(2n+\mu-3)}{(2n+\mu-1)^2}g-1, \frac{\deg \mathcal{K}_X \otimes \mathcal{M}^{-1}}{3} \right\},$$

then \mathcal{M} is composed with the simple n -fold covering morphism ϕ . The condition $\deg \mathcal{M} \leq g-1$ gives

$$\frac{\deg \mathcal{K}_X \otimes \mathcal{M}^{-1}}{3} \geq \frac{g}{3} - 1 \geq \frac{2(2n+\mu-3)}{(2n+\mu-1)^2}g-1,$$

whence the result follows. \square

Applying Lemma 4.6 to a simple multiple covering X of a smooth plane curve, we not only find the Clifford index of X but also characterize the line bundles computing the Clifford index of X .

Proposition 4.7. *Assume that a smooth curve X of genus g admits a simple n -fold covering morphism $\phi : X \rightarrow Y$ for a smooth plane curve Y of degree d with $d > \frac{4n^2+2}{3}$ and $g \geq ng(Y) + n(n-1)d + 2n^2(n-1)$. Let \mathcal{M} be a line bundle computing the Clifford index of X with $\deg \mathcal{M} \leq g-1$. Then*

$$\begin{aligned} \mathcal{M} &\simeq \phi^* \mathcal{E}(-Q), \quad Q \in Y \quad \text{in case } n \geq 3 \\ \mathcal{M} &\simeq \phi^* \mathcal{E} \quad \text{or } \mathcal{L} \simeq \phi^* \mathcal{E}(-Q), \quad Q \in Y \quad \text{in case } n = 2, \end{aligned}$$

where $\mathcal{E} := \mathcal{O}_Y(1)$. Specifically, we have $\text{Cliff}(X) = nd - n - 2$.

Proof. Assume that $\mathcal{M} = \phi^* \mathcal{N}$ on X is composed with ϕ . Since $\text{Cliff}(\mathcal{N}) > d-4$ for a line bundle \mathcal{N} on Y with $\mathcal{N} \neq \mathcal{E}$, we have $\text{Cliff}(\mathcal{M}) > nd-4$ in case $\mathcal{N} \neq \mathcal{E}$ or $\mathcal{E}(-Q)$. On the one hand,

$$\text{Cliff}(\phi^* \mathcal{E}) = nd - 4 \quad \text{and} \quad \text{Cliff}(\phi^* \mathcal{E}(-Q)) = nd - n - 2$$

for $Q \in Y$. Thus Lemma 4.6 implies the theorem if we verify the following claim, where $\gamma := g(Y)$ and $\mu := \lceil \frac{2n(n-1)\gamma}{g-n\gamma} \rceil$.

Claim: $nd - n - 2 < \min\left\{ \frac{g-n\gamma}{n-1} - 3, \frac{2(2n+\mu-3)}{(2n+\mu-1)^2}g-1 \right\}$.

Proof of Claim. Since the condition $g \geq n\gamma + n(n-1)d + 2n^2(n-1)$ gives $nd - n - 2 < \frac{g-n\gamma}{n-1} - 3$, it suffices to show that $nd - n - 2 < \frac{2(2n+\mu-3)}{(2n+\mu-1)^2}g - 1$. To prove this, we note the inequality

$$\mu \leq d - 2n - 1.$$

This is also given by $g \geq n\gamma + n(n-1)d + 2n^2(n-1)$ and $d > \frac{4n^2+2}{3}$ which imply

$$2\gamma \frac{n(n-1)}{g-n\gamma} - (d-2n) \leq \frac{(d-1)(d-2)}{d+2n} - (d-2n) \leq \frac{-3d+4n^2+2}{d+2n} < 0,$$

for $\gamma = \frac{(d-1)(d-2)}{2}$.

Now, we will divide the proof into the following three cases:

$$(1) \ n = 2, \quad (2) \ n \geq 3 \text{ and } \mu > 0, \quad (3) \ n \geq 3 \text{ and } \mu = 0.$$

(1) Assume that $n = 2$. In case $\mu = 0$, the inequality $2d - 4 < \frac{2(\mu+1)}{(\mu+3)^2}g - 1$ trivially comes from $g \geq n\gamma + n(n-1)d + 2n^2(n-1) = d^2 - d + 10$. Thus we assume $\mu \geq 1$. According to the inequality $\mu \leq d - 2n - 1 = d - 5$ and the condition $g \geq d^2 - d + 10$, we obtain

$$\begin{aligned} & \left\{ \frac{2(\mu+1)}{(\mu+3)^2}g - 1 \right\} - \{2d - 4\} \\ & \geq \frac{1}{(\mu+3)^2} \{(\mu+1)(2d^2 - 2d + 20) - (2d-3)(\mu+3)^2\} \\ & = \frac{1}{(\mu+3)^2} \{-(2d-3)\mu^2 + (2d-3)(d-5)\mu - (d-23)\mu + 2d^2 - 20d + 47\} \\ & \geq \frac{1}{(\mu+3)^2} \{-(d-23)\mu + 2d^2 - 20d + 47\} \\ & = \frac{1}{(\mu+3)^2} \{-(d-23)\mu + d(d-5) + d^2 - 15d + 47\} \\ & \geq \frac{1}{(\mu+3)^2} \{-(d-23)\mu + d\mu + d^2 - 15d + 47\} > 0. \end{aligned}$$

This proves the claim for $n = 2$.

(2) Assume that $n \geq 3$ and $\mu > 0$. Since $g \geq n\gamma + n(n-1)d + 2n^2(n-1)$ and $2\gamma = (d-1)(d-2)$, we get

$$2g > n(d^2 - 3d) + 2n(n-1)d = nd(d+2n-5) \geq nd(d+1).$$

This gives

$$\begin{aligned}
& \left\{ \frac{2(2n + \mu - 3)}{(2n + \mu - 1)^2} g - 1 \right\} - \{nd - n - 2\} \\
& > \frac{1}{(2n + \mu - 1)^2} \{(2n + \mu - 3)nd(d + 1) - (nd - n - 1)(2n + \mu - 1)^2\} \\
& > \frac{nd}{(2n + \mu - 1)^2} \{(2n + \mu - 3)(d + 1) - (2n + \mu - 1)^2\} \\
& \geq \frac{nd}{(2n + \mu - 1)^2} \{2n + \mu - 7\} \geq 0,
\end{aligned}$$

for $\mu > 0$, $\mu \leq d - 2n - 1$. Thus the claim is verified in case $n = 3$ and $\mu > 0$.

(3) Assume that $n \geq 3$ and $\mu = 0$. Then we have

$$\begin{aligned}
& \left\{ \frac{2(2n + \mu - 3)}{(2n + \mu - 1)^2} g - 1 \right\} - \{nd - n - 2\} \\
& = \frac{2(2n - 3)}{(2n - 1)^2} g - nd + n + 1 > \frac{n(2n - 3)}{(2n - 1)^2} (d - 1)(d - 2) - nd \\
& \geq n \left\{ \frac{4n^2(2n - 3)}{3(2n - 1)^2} (d - 2) - d \right\} \geq n \left\{ \frac{4}{3} (d - 2) - d \right\} > 0,
\end{aligned}$$

since $2g > 2n\gamma = n(d - 1)(d - 2)$, $d - 1 \geq \frac{4n^2}{3}$, and $\frac{n^2(2n-3)}{(2n-1)^2} \geq 1$ for $n \geq 3$. This proves the claim in case $n \geq 3$ and $\mu = 0$. Thus we complete the proof of the theorem. \square

From this we obtain the following theorem which demonstrates the sharpness of Conjecture 4.1 on simple multiple coverings of smooth plane curves. As mentioned in §3, the condition $g \geq ng(Y) + 2n(n - 1)d$ in the theorem is not so strong for $d \gg n$, since any multiple covering X of a smooth curve Y satisfies the inequality $g(X) \geq ng(Y) - n$ by the Riemann-Hurwitz Formula.

Theorem 4.8. *Let a smooth curve X of genus g be a simple n -fold covering of a smooth plane curve Y of degree d with $d > \frac{4n^2+2}{3}$ and $g \geq ng(Y) + 2n(n - 1)d$. For any $p \leq nd - 4n - 2$, the curve X carries extremal line bundles for Green-Lazarsfeld's conjecture on (N_p) which are given by $\mathcal{L} \simeq \mathcal{K}_X - \phi^*(g_d^2(-\tilde{Q})) + \xi_{p+4}^0$ for some $\tilde{Q} \in Y$ and $\xi_{p+4}^0 \in X^{(p+4)}$ satisfying that $\deg(\phi^*(H - \tilde{Q}), \xi_{p+4}^0) \leq 1$ for any $H \in g_d^2$ with $\tilde{Q} \leq H$ and $\deg(\phi^*(H), \xi_{p+4}^0) \leq n + 1$ for any $H \in g_d^2$.*

Proof. Assume that $\mathcal{L} \simeq \mathcal{K}_X - \phi^*(g_d^2(-\tilde{Q})) + \xi_{p+4}^0$ for some $\tilde{Q} \in Y$ and $\xi_{p+4}^0 \in X^{(p+4)}$ satisfying the conditions of the theorem. Note that the hypotheses $d > \frac{4n^2+2}{3}$ and $g \geq ng(Y) + 2n(n - 1)d$ imply both $g \geq ng(Y) + n(n - 1)d + 2n^2(n - 1)$ and $\lceil \frac{g - ng(Y)}{n - 1} \rceil - nd + 3 \geq nd - 5n + 3$. Hence, by Proposition 4.7 and Theorem 3.10,

$\text{Cliff}(X)$ is equal to $nd-n-2$ and \mathcal{L} is minimally presented by $\mathcal{K}_X - g_{nd-n-1}^0 + \xi_{p+3}^0$, where $g_{nd-n-1}^0 := \phi^*(g_d^2(-\tilde{Q})) - P$, $\xi_{p+3}^0 := \xi_{p+4}^0 - P$, $P \leq \xi_{p+4}^0$. It means that \mathcal{L} is a very ample line bundle with $\deg \mathcal{L} = 2g + p - 2h^1(X, \mathcal{L}) - \text{Cliff}(X)$ and the curve $\varphi_{\mathcal{L}}(X)$ has no $(p+2)$ -secant p -planes.

Assume that property (N_p) holds for \mathcal{L} . Then the line bundle \mathcal{M} given by

$$\mathcal{M} \simeq \mathcal{K}_X - g_{nd-n-1}^0 + \xi_3^0, \quad \xi_3^0 \leq \xi_{p+3}^0,$$

is normally generated by Theorem 1 in [6]. However, it cannot occur by the same reason in the proof of Theorem 4.5, since we have $h^0(X, \mathcal{M}) - h^0(X, \mathcal{M}(-(P + \xi_3^0))) = 2$ by $g_{nd-n-1}^0 := \phi^*(g_d^2(-\tilde{Q})) - P$. Thus \mathcal{L} does not satisfy property (N_p) . Consequently, \mathcal{L} is an extremal line bundle for Green-Lazarsfeld's conjecture on (N_p) . \square

Remark 4.9. Let X admit a morphism $\phi : X \rightarrow Y \subset \mathbb{P}^2$ for a smooth plane curve Y of degree d with $n := \deg \phi \leq 2$. Assume that $0 \leq p \leq nd - 5n - 3$ and $g \geq 2g(Y) + 2d + 8$ for $n = 2$. Then, X carries another type of extremal line bundles for Green-Lazarsfeld's conjecture on (N_p) as follows. Choose a ξ_{p+6}^0 on X satisfying (i) the points of ξ_{p+6}^0 are distinct and map to distinct points of Y , (ii) there is a $\xi_6^0 \leq \xi_{p+6}^0$ such that points of $\phi(\xi_6^0)$ lie on a conic but has no four collinear. Then, $\mathcal{L} \simeq \mathcal{K}_X - \phi^*g_d^2 + \xi_{p+6}^0$ is an extremal line bundle for Green-Lazarsfeld's conjecture on (N_p) . This can be shown by the same way as Theorem 4.8; (1) $\varphi_{\mathcal{L}}(X)$ has no $(p+2)$ -secant p -plane since Theorem 3.10 implies that \mathcal{L} is minimally presented by $\mathcal{K}_X - g_{nd-2}^0 + \xi_{p+4}^0$ with $g_{nd-2}^0 := \phi^*(g_d^2) - (P_1 + P_2)$ and $\xi_{p+4}^0 := \xi_{p+6}^0 - (P_1 + P_2) \geq 0$, (2) \mathcal{L} does not satisfy property (N_p) by Theorem 1 in [6] since $\mathcal{L}(-\xi_p^0) \simeq \mathcal{K}_X - \phi^*g_d^2 + \xi_6^0$ fails to be normally generated as (2.5) Remark in [7], where $\xi_p^0 := \xi_{p+6}^0 - \xi_6^0$.

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